Application of MS-Excel "Solver" to Non-linear Beam Analysis

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1. Introduction

Spreadsheet software in current personal computers is high performance and the software has enough functions for application to engineering problems. I developed a method to solve non-linear structural problems by optimization function "Solver" of MS-Excel.

2. MS-Excel "Solver"

The most popular spreadsheet software, MS-Excel has an add-in tool called "Solver" which performs numerical optimization. "Solver" finds parameters to optimize "objective value" with multiple constraints.

We can use "Solver" to solve structural problems applying principle of total potential energy or principle of total complimentary energy.

3. Equations – Non-linear Beam Element

Beam structure is discretized to elements as finite element method (FEM). Then, total potential energy is calculated. "Shape function" is the same as FEM formulation. This means that the formulation is the same as FEM, but we don't need to perform variational operation. We just need to calculate strain energy of each beam element and sum up the strain energy for all the elements.

Following formulation show strain energy of 2-dimentional beam element.

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(1) Beam Element

Assume that properties of a beam element as follows.

- Uniform cross section in a element. Section Area, A_e, Moment of Inertia, I_e, Length, L_e
- Young's Modulus, E_e

As shown in figure 1, following symbols are used.

x, y deflection and rotation at Grid 1 in **element coordinates before deflection**: $(u_{e1}, v_{e1}, \theta_{e1})$

x, y deflection and rotation at Grid 2 in **element coordinates before deflection**: $(u_{e2}, v_{e2}, \theta_{e2})$

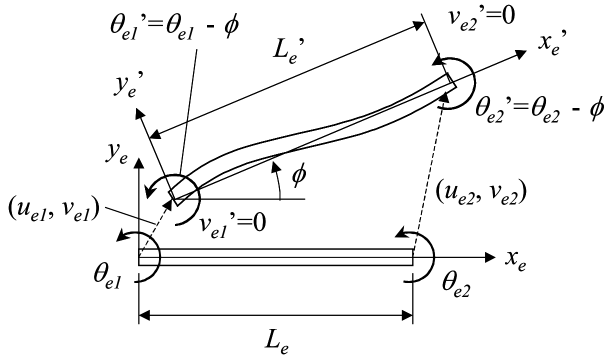

Figure 1. Beam Element

(2) Bending

Deflection of element, v_e is expressed as follows.

$$v_e(x_e)=ax_e^{\ 3}+bx_e^{\ 2}+cx_e+d\,,\quad \frac{dv_e}{dx_e}=3ax_e^{\ 2}+2bx_e+c\,,\quad \frac{d^2v_e}{dx_e^{\ 2}}=6ax_e+2b$$

at $x_e=0$, $v_{e1}=d$, $\left(\frac{dv_e}{dx_e}\right)_{x_e=0}=\theta_{e1}=c$

at $x_e=L_e$, $v_{e2}=aL_e^{\ 3}+bL_e^{\ 2}+cL_e+d$, $\left(\frac{dv_e}{dx_e}\right)_{x_e=L_e}=\theta_{e2}=3aL_e^{\ 2}+2bL_e$

Solving the equations,

$$a=\frac{\theta_{e1}+\theta_{e2}}{L_e^{\ 2}}-\frac{2(v_{e2}-v_{e1})}{L_e^{\ 3}},\qquad b=-\frac{2\theta_{e1}+\theta_{e2}}{L_e}+\frac{3(v_{e2}-v_{e1})}{L_e^{\ 2}}$$

$$c=\theta_{e1},\qquad d=v_{e1}$$

Strain energy for bending is expressed as follows.

$$U_b=\frac{E_eI_e}{2}\int_0^{L_e}\left(\frac{d^2v_e}{dx_e^{\ 2}}\right)^2dx_e=\frac{E_eI_e}{2}\int_0^{L_e}(6ax_e+2b)^2dx_e=E_eI_eL_e\left(6a^2L_e^{\ 2}+6abL_e+2b^2\right)$$

To calculate shear force and bending moment, use following beam equations.

Shear force and bending moment at Grid 1, (V_1, M_1)

Shear force and bending moment at Grid 2, (V_2, M_2)

Equation of beam: $E_eI_e\frac{d^2v_e}{dx_e^{\ 2}}=-M_e$

$$V_1=E_eI_ec\,,\ \ M_1=-2E_eI_eb$$
$$V_2=-V_1\,,\quad M_2=-M_1+V_1L_e$$

(3) Extension

Length after deflection L_e' is expressed as follows.

$$L_e'=\sqrt{(L_e+u_{e2}-u_{e1})^2+(v_{e2}-v_{e1})^2}$$

Extension ΔL_e is,

$$\Delta L_e=\sqrt{(L_e+u_{e2}-u_{e1})^2+(v_{e2}-v_{e1})^2}-L_e$$

Strain energy U_e is,

$$U_e=\frac{EA}{2L_e}\Delta L^2$$

To calculate axial force at Grid 1, P_{e1} and Grid 2, P_{e2} , use following equations.

$$P_{e1} = -EA\frac{\Delta L}{L_e}, \quad P_{e2} = -P_{e1}$$

(4) Consideration of Geometrical Non-linearity in Coordinate System after Deformation

Element coordinate system after deflection is considered as shown in figure 1. Element defections in element coordinate system after deflection, v_{e1}, v_{e2} become zero and strain energy for bending of the element is expressed as follows.

$$a = \frac{\theta_{e1}' + \theta_{e2}'}{L_e^{\,2}}, \qquad b = -\frac{2\theta_{e1}' + \theta_{e2}'}{L_e}, \qquad c = \theta_{e1}', \qquad d = 0$$

where, $\theta_{e1}' = \theta_{e1} - \phi$, $\theta_{e2}' = \theta_{e2} - \phi$

ϕ is angle between element coordinate after and before deflection.

Then, strain energy for bending (geometrical non-linearity is considered), U_b is expressed in the following equation.

$$U_b = E_e I_e L_e \left(6a^2 L_e^{\,2} + 6abL_e + 2b^2\right)$$

(5) Total Potential Energy

Total potential energy of the element, U_{total} is expressed as the following equation.

$$U_{total} = \sum_{all\ elements}\left(U_e + U_b\right) - \sum_{all\ external\ forces}\left(P_x u_x + P_y v_x + M_z \theta_z\right)$$

Using deflections and rotations at all grids as parameters, minimize total potential energy, U_{total} by "Solver" of MS-Excel, then you will have the solution.

4. Example Problem – Elastica

Post-buckling deflection of a simply supported beam under axial force is analyzed. This problem is a typical geometrically non-linear problem known as "Elastica". Analytical solution is available for "Elastica" and the result of the present method is compared with the analytical solution.

A beam with uniform cross section is divided to 20 elements as shown in figure 2. A template of MS-Excel to calculate total potential energy was developed. Then, the total potential energy was minimized by "Solver".

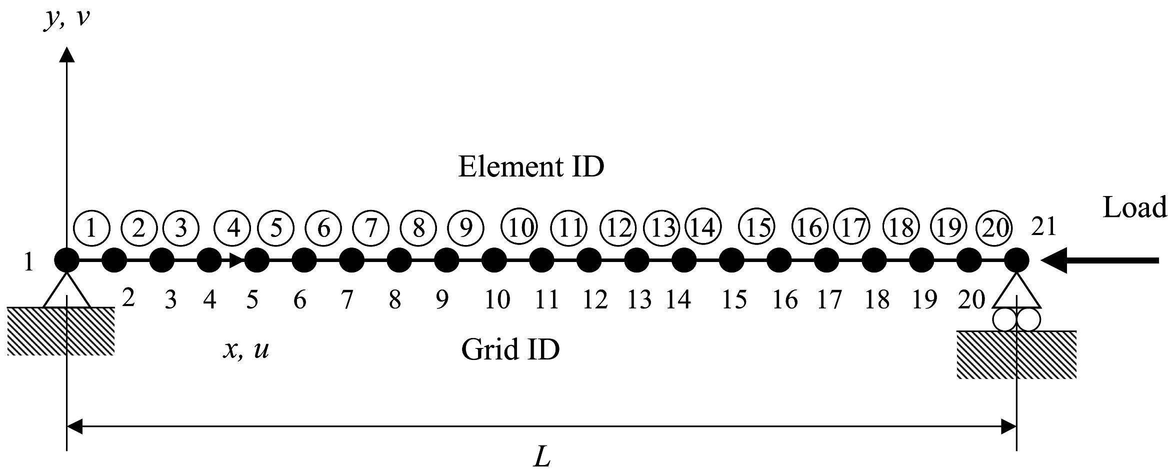

Figure 2. Model for Elastica

Figure 3 shows deflection shape for various axial forces.

Figure 4 shows relationship between center deflection and axial force. Center deflection and axial force are normalized with beam length, L and buckling load, P_{cr} , respectively. Exact analytical solution is shown in the figure for comparison (reference [1]).

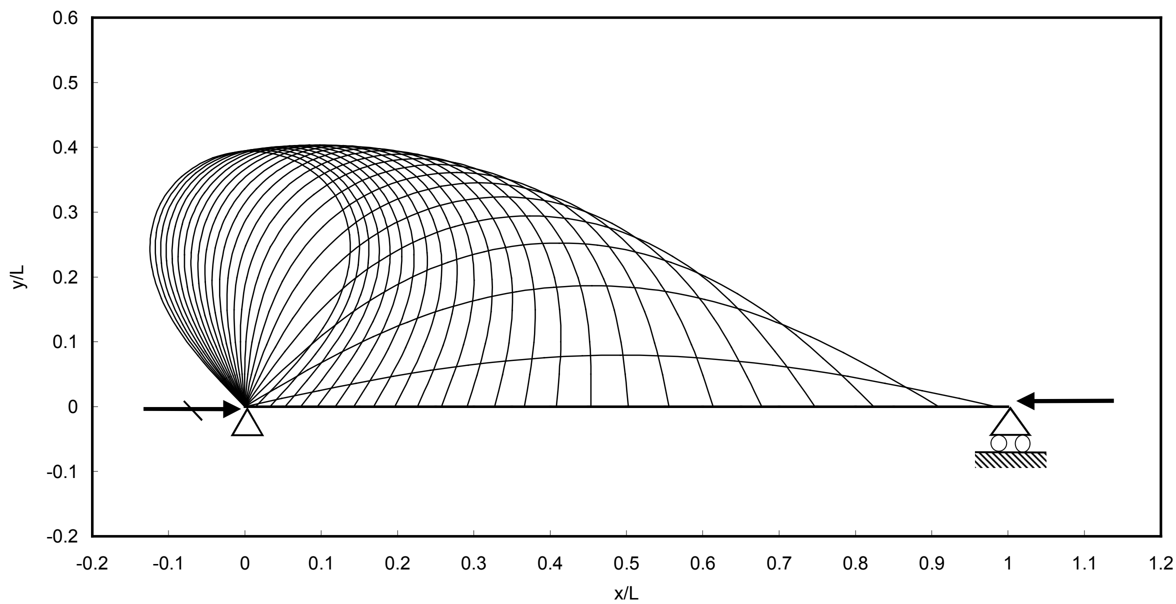

Figure 3. Deflection Shape

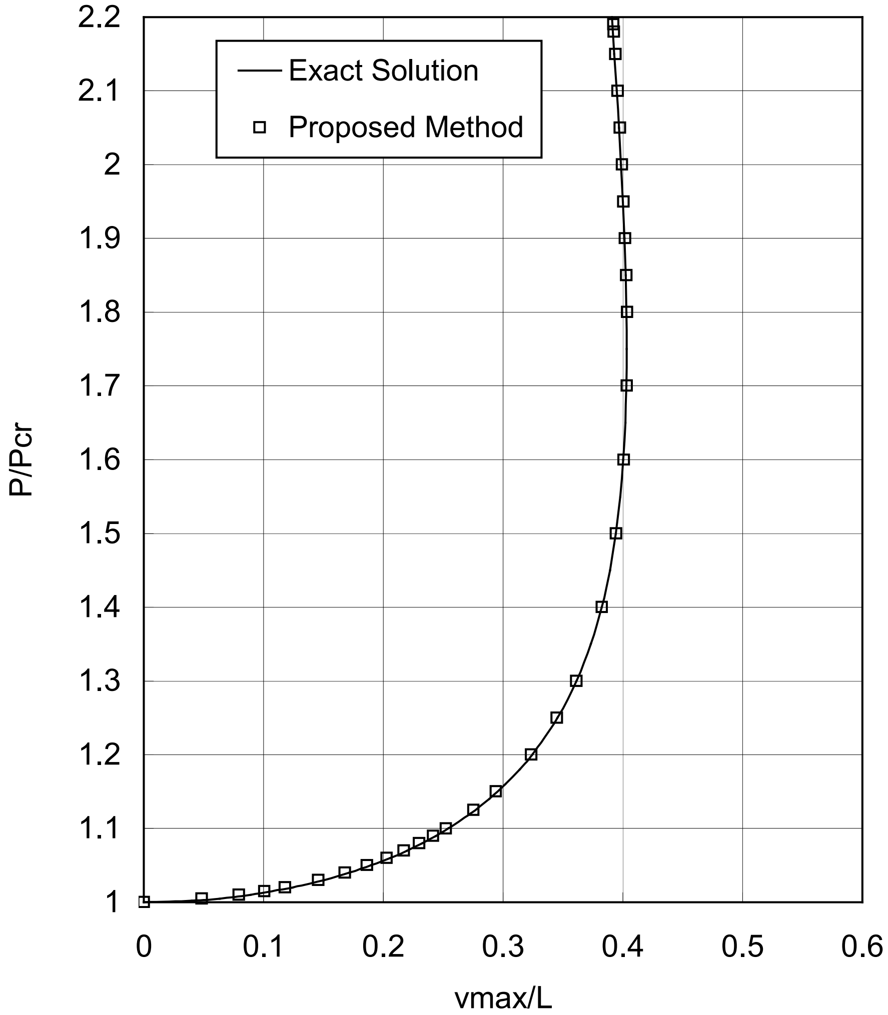

Figure 4. Relationship between Center Deflection and Axial Force

5. Other Applications

The method I introduced in this paper can be applied to following problems in aircraft structural analysis. I developed the method 10 years ago and have applied to many problems of actual aircraft structural analysis.

- Truss (linear, or geometrically non-linear)
- Beam and Ramen (linear, or geometrically non-linear)
- Bolted Joint (linear)
- Beam Column (geometrically non-linear)
- 2-dimensional Elastic Problems (linear, or geometrically non-linear)
- Shear Field Problems (linear)

6. Conclusions

The method presented in this paper has following features and I expect that the method will be widely used in the aircraft industry.

- Straight forward method. Special technique and tedious deviation of equations are not necessary to solve non-linear problems.
- No special programs are required. General purpose spreadsheet software is used.
- Applicable to actual aircraft structural analysis.
- Useful to education of energy method.

7. References

[1] C. L. Dym and I. H. Shames, "Solid Mechanics, A Variational Approach"